22107207

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

**MATHEMATICS**
**HIGHER LEVEL**
**PAPER 3 – DISCRETE MATHEMATICS**

Thursday 20 May 2010 (afternoon)

1 hour

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INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

*Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.*

**1.** *[Maximum mark: 14]*

(a) (i) One version of Fermat's little theorem states that, under certain conditions,

$$a^{p-1} \equiv 1 \pmod{p}.$$

Show that this result is not valid when $a = 4$, $p = 9$ and state which condition is not satisfied.

(ii) Given that $5^{64} \equiv n \pmod{7}$, where $0 \le n \le 6$, find the value of $n$. *[8 marks]*

(b) Find the general solution to the simultaneous congruences

$$x \equiv 3 \pmod{4}$$
$$3x \equiv 2 \pmod{5}.$$ *[6 marks]*

**2.** *[Maximum mark: 9]*

A graph $G$ with vertices A, B, C, D, E has the following cost adjacency matrix.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| A | – | 12 | 10 | 17 | 19 |
| B | 12 | – | 13 | 20 | 11 |
| C | 10 | 13 | – | 16 | 14 |
| D | 17 | 20 | 16 | – | 15 |
| E | 19 | 11 | 14 | 15 | – |

(a) (i) Use Kruskal's algorithm to find and draw the minimum spanning tree for $G$.

(ii) The graph $H$ is formed from $G$ by removing the vertex D and all the edges connected to D. Draw the minimum spanning tree for $H$ and use it to find a lower bound for the travelling salesman problem for $G$. *[7 marks]*

(b) Show that 80 is an upper bound for this travelling salesman problem. *[2 marks]*

**3.** *[Maximum mark: 12]*

The positive integer $N$ is expressed in base 9 as $(a_n a_{n-1} \ldots a_0)_9$.

(a) Show that $N$ is divisible by 3 if the least significant digit, $a_0$, is divisible by 3. *[3 marks]*

(b) Show that $N$ is divisible by 2 if the sum of its digits is even. *[3 marks]*

(c) Without using a conversion to base 10, determine whether or not $(464860583)_9$ is divisible by 12. *[6 marks]*

**4.** *[Maximum mark: 18]*

(a) Show that, for a connected planar graph,

$$v + f - e = 2 .$$ *[7 marks]*

(b) Assuming that $v \geq 3$, explain why, for a simple connected planar graph, $3f \leq 2e$ and hence deduce that $e \leq 3v - 6$. *[4 marks]*

(c) The graph $G$ and its complement $G'$ are simple connected graphs, each having 12 vertices. Show that $G$ and $G'$ cannot both be planar. *[7 marks]*

**5.** *[Maximum mark: 7]*

Given that $a, b, c, d \in \mathbb{Z}$, show that

$$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d) \equiv 0 \pmod 3 .$$